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Research Article

Truncation error bounds of branched continued fraction expansions of special ratios of Horn's hypergeometric functions H_4

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ABSTRACT. The paper considers the branched continued fraction extensions of special ratios of Horn's hypergeometric functions H_4 with real parameters and variables. Truncation error bounds are established for such expansions with certain conditions on their coefficients. Some domains of analytical continuation of the above-mentioned special ratios are also established using the PF method (based on the so-called property of fork for approximants of a branched continued fraction).

Keywords: Hypergeometric function, branched continued fraction, approximation by rational functions, rate of convergence, analytic continuation.

2020 Mathematics Subject Classification: 33C65, 32A17, 32A10, 41A25, 30B40.

1. INTRODUCTION

The study of special functions has been and remains relevant for several centuries due to their practical application in many fields of science [6, 16, 20, 22].

The paper investigates branched continued fraction expansions of the ratios of hypergeometric functions of two variables [15, 16, 17]. An overview of such expansions was described in [3]. The application of branched continued fraction expansions to the approximation of special functions represented by double hypergeometric series were considered in [1, 2, 7, 14]. In this paper, we continue our study of the expansions of special ratios of the Horn's hypergeometric functions H_4 [2, 9, 10, 11, 12, 8].

Recall that the function H_4 is defined as (see [15, Section 5.7] and [17])

$$H_4(\alpha, \beta; \gamma, \delta; \mathbf{z}) = \sum_{r,s=0}^{\infty} \frac{(\alpha)_{2r+s}(\beta)_s}{(\gamma)_r(\delta)_s} \frac{z_1^r}{r!} \frac{z_2^s}{s!},$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ herewith $\gamma, \delta \notin \{0, -1, -2, \dots\}$, $\mathbf{z} = (z_1, z_2) \in \mathfrak{D}_{p,q}$,

$$\mathfrak{D}_{p,q} = \left\{ \mathbf{z} \in \mathbb{C}^2 : \ |z_1| < p, \ |z_2| < q \right\}, \ p > 0, \ q > 0, \ 4p = (q-1)^2, \ q \neq 1, \tag{1.1}$$

$(\xi)_k$ is the Pochhammer symbol, $(\xi)_k = \Gamma(\xi + k)/\Gamma(\xi)$, $\Gamma(z)$ is the gamma function.

Note that some relations for the Horn's hypergeometric function H_4 were obtained in [5, 19, 21], including differentiation and integration formulas, series for special values of parameters and variables, and some generating functions for various special functions in terms of this function.

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As a special case of [1, Theorem 1], we have the following theorem:

Theorem A. *The ratios*

$$\frac{H_4(\alpha,\beta;\gamma,\beta;\mathbf{z})}{H_4(\alpha+1,\beta;\gamma+1,\beta;\mathbf{z})}, \tag{1.2}$$

$$\frac{H_4(\alpha,\delta+1;\gamma,\delta;\mathbf{z})}{H_4(\alpha+1,\delta+1;\gamma,\delta+1;\mathbf{z})}, \tag{1.3}$$

and

$$\frac{H_4(\alpha,\delta+1;\gamma,\delta;\mathbf{z})}{H_4(\alpha,\delta+2;\gamma,\delta+1;\mathbf{z})} \tag{1.4}$$

have formal branched continued fraction expansions

$$1-z_2-\cfrac{a_1z_1}{1-z_2-\cfrac{a_2z_1}{1-z_2-\cfrac{a_3z_1}{1-\ddots}}}, \tag{1.5}$$

where

$$a_k=\frac{(2\gamma-\alpha+k-1)(\alpha+k)}{(\gamma+k-1)(\gamma+k)},\ k\geq 1,$$

$$1-\frac{\delta-\alpha}{\delta}z_2-\cfrac{b_1z_1}{1-z_2-\cfrac{b_2z_1}{1-z_2-\cfrac{b_3z_1}{1-\ddots}}}, \tag{1.6}$$

where

$$b_1=\frac{2(\alpha+1)}{\gamma},\ b_k=\frac{(2\gamma-\alpha+k-3)(\alpha+k)}{(\gamma+k-2)(\gamma+k-1)},\ k\geq 2, \tag{1.7}$$

and

$$1+\cfrac{d_0z_2}{1-d_1z_2-\cfrac{c_1z_1}{1-d_2z_2-\cfrac{c_2z_1}{1-\ddots}}}, \tag{1.8}$$

where

$$\begin{aligned} d_0&=\frac{\alpha}{\delta(\delta+1)},\ d_1=1-\frac{\alpha}{\delta+1},\ c_1=\frac{2(\alpha+1)}{\gamma},\\ d_k&=1,\ c_k=\frac{(2\gamma-\alpha+k-3)(\alpha+k)}{(\gamma+k-2)(\gamma+k-1)},\ k\geq 2, \end{aligned} \tag{1.9}$$

respectively.

Different domains of analytical continuation of ratio (1.2) by branched continued fraction (1.5) were established in [1, 12] by the PC method (based on the so-called principle of correspondence between a formal double power series and a branched continued fraction). Some domains of convergence of the expansion (1.6) and (1.8) was studied in [11] and [10], respectively. Truncation error bound for expansion (1.5) with certain conditions on real parameters was established in [8]. Here, a new domain of analytical continuation of (1.2) was also established by the PF method (based on the so-called property of fork for approximants of a branched continued fraction).

As in the theory of continued fractions [18], we have the following definition:

Definition A. *Let for each $\mathbf{z}\in\mathfrak{D}$, $\mathfrak{D}\subset\mathbb{C}^2$, the branched continued fraction*

$$v_0(\mathbf{z})+\sum_{i_1=1}^{2}\frac{u_{i_1}(\mathbf{z})}{v_{i_1}(\mathbf{z})+\sum\limits_{i_2=1}^{2}\dfrac{u_{i_1,i_2}(\mathbf{z})}{v_{i_1,i_2}(\mathbf{z})+\ddots}}\tag{1.10}$$

converges to the finite value $f(\mathbf{z})$, where $v_0(\mathbf{z})$, $u_{i_1}(\mathbf{z})$, $v_{i_1}(\mathbf{z})$, $u_{i_1,i_2}(\mathbf{z})$, $v_{i_1,i_2}(\mathbf{z}),\ldots$ are functions of $\mathbf{z}$. Let $f_k(\mathbf{z})$ denote the kth approximant of (1.10), $k\ge1$. Then

$$f(\mathbf{z})-f_k(\mathbf{z})$$

is called the truncation error of the kth approximant, and

$$|f(\mathbf{z})-f_k(\mathbf{z})|\le C_k(\mathbf{z})$$

is called a priori bound (or truncation error bound), where $C_k(\mathbf{z})\ge0$ and $C_k(\mathbf{z})\to0$ as $k\to+\infty$ for all $\mathbf{z}\in\mathfrak{D}$.

In this paper, we obtain truncation error bounds for branched continued fractions (1.6) and (1.8), and also establish domains of analytical continuation of the functions (1.3) and (1.4).

2. Main results

The following theorem holds.

Theorem 2.1. *Suppose that α,γ and δ are real numbers such that*

$$0<d_1\le\kappa,\ 0<c_k\le\kappa,\ k\ge1,\tag{2.11}$$

where d_1, c_k, $k\ge1$, are defined by (1.9), $\gamma\notin\{0,-1,-2,\ldots\}$, $\delta\notin\{0,-1,-2,\ldots\}$, κ is a positive number. Then:

(A) *The branched continued fraction (1.8) converges to a finite value $f(\mathbf{z})$ for each $\mathbf{z}\in\mathfrak{D}_\eta$, where*

$$\mathfrak{D}_\eta=\left\{\mathbf{z}\in\mathbb{R}^2:\ z_1\le0,\ z_2\le\eta\right\},\ 0<\eta<\min\left\{\frac{1+\delta}{1-\alpha+\delta},\ 1\right\}.\tag{2.12}$$

(B) *The convergence is uniformly on every compact subset of $\operatorname{Int}(\mathfrak{D}_\eta)$, and $f(\mathbf{z})$ is analytic on $\operatorname{Int}(\mathfrak{D}_\eta)$.*

(C) *For each $\mathbf{z}\in\mathfrak{D}_\eta$ and for $n\ge3$*

$$|f(\mathbf{z})-f_n(\mathbf{z})|\le\frac{|d_0||z_2|(|z_2|(1-z_2)+\kappa|z_1|)(1-z_2)^{-2}(1-z_2+\kappa|z_1|)^{-2}(\kappa|z_1|)^{n-1}}{(1-d_1z_2)((1-d_1z_2)(1-z_2)+\kappa|z_1|)((1-z_2)^2+\kappa|z_1|)^{n-3}},$$

where $f_n(\mathbf{z})$ is the nth approximant of (1.8).

(D) *The function $f(\mathbf{z})$ is an analytic continuation of (1.4) in $\operatorname{Int}(\mathfrak{D}_\eta)$.*

Note that the conditions (2.11) are satisfied if $0<\alpha<\delta+1$ and $\alpha<2\gamma-1$.

Proof of the Theorem 2.1. Let us use the idea of the proving Theorem 1 from [8]. First we set

$$U_n^{(n)}(\mathbf{z})=1,\ n\ge1,\tag{2.13}$$

and

$$U_k^{(n)}(\mathbf{z}) = 1 - d_k z_2 - \cfrac{c_k z_1}{1 - d_{k+1} z_2 - \cfrac{c_{k+1} z_1}{1 - \ddots - d_{n-2} z_2 - \cfrac{c_{n-2} z_1}{1 - d_{n-1} z_2 - c_{n-1} z_1}}},$$

where $1 \leq k \leq n-1,\ n \geq 2$. Then

$$U_k^{(n)}(\mathbf{z}) = 1 - d_k z_2 - \frac{c_k z_1}{U_{k+1}^{(n)}(\mathbf{z})},\ 1 \leq k \leq n-1,\ n \geq 2, \tag{2.14}$$

and

$$f_n(\mathbf{z}) = 1 + \frac{d_0 z_2}{U_1^{(n)}(\mathbf{z})}, \quad n \geq 1.$$

Now, we will proof (A). Let $\mathbf{z}$ be an arbitrary fixed point in (2.12). From (2.11) it follows that the coefficients d_1, c_k, $k \geq 1$, are positive real numbers. Using inequalities from (2.12) and the relations (2.13) and (2.14), for any $n \geq 2$ we obtain

$$\begin{aligned} U_1^{(n)}(\mathbf{z}) &= 1 - d_1 z_2 - \frac{c_1 z_1}{U_2^{(n)}(\mathbf{z})} \\ &\geq 1 - d_1 z_2 \\ &\geq 1 - d_1 \eta \\ &> 0, \end{aligned}$$

and for arbitraries $n \geq 3$ and $2 \leq k \leq n-1$, we get

$$\begin{aligned} U_k^{(n)}(\mathbf{z}) &= 1 - d_k z_2 - \frac{c_k z_1}{U_{k+1}^{(n)}(\mathbf{z})} \\ &\geq 1 - z_2 \\ &\geq 1 - \eta \\ &> 0. \end{aligned}$$

This allows us to use the well-known formula form [4, p. 28]. Therefore, for $n \geq 2$ and $k \geq 1$

$$f_{n+k}(\mathbf{z}) - f_n(\mathbf{z}) = d_0 z_1^{n-1} z_2 \left(z_2 + \frac{c_n z_1}{U_{n+1}^{(n+k)}(\mathbf{z})} \right) \prod_{r=1}^{n-1} \frac{c_r}{U_r^{(n+k)}(\mathbf{z}) U_r^{(n)}(\mathbf{z})}$$

or the same

$$\begin{aligned} f_{n+k}(\mathbf{z}) - f_n(\mathbf{z}) = &\frac{d_0 z_1^{n-1} z_2}{U_1^{(q)}(\mathbf{z}) U_n^{(n+k)}(\mathbf{z})} \left(z_2 + \frac{c_n z_1}{U_{n+1}^{(n+k)}(\mathbf{z})} \right) \\ &\times \prod_{r=1}^{[(n-1)/2]} \frac{c_{2r-1}}{U_{2r-1}^{(p)}(\mathbf{z}) U_{2r}^{(p)}(\mathbf{z})} \prod_{r=1}^{[(n-2)/2]} \frac{c_{2r}}{U_{2r}^{(q)}(\mathbf{z}) U_{2r+1}^{(q)}(\mathbf{z})}, \end{aligned}$$

where $[.]$ denote integer part, $q = n+k$, $p = n$, if $n = 2s$, and $q = n$, $p = n+k$, if $n = 2s-1$, $s \geq 1$.

Now, for arbitraries $m \geq 2$ and $k \geq 2$ we have

$$\frac{d_0 z_2}{U_1^{(m)}(\mathbf{z})} \leq \frac{|d_0||z_2|}{1 - d_1 z_2}, \frac{1}{U_m^{(m+k)}(\mathbf{z})} \left(z_2 + \frac{c_m z_1}{U_{m+1}^{(m+k)}(\mathbf{z})} \right) \leq \frac{1}{1 - z_2} \left(|z_2| + \frac{\kappa |z_1|}{1 - z_2} \right),$$

for any $m \geq 2$ we obtain

$$\frac{c_1 z_1}{U_1^{(m+1)}(\mathbf{z})U_2^{(m+1)}(\mathbf{z})} = \frac{\dfrac{c_1 z_1}{U_2^{(m+1)}(\mathbf{z})}}{1 - d_1 z_2 - \dfrac{c_1 z_1}{U_2^{(m+1)}(\mathbf{z})}}$$
$$\leq \frac{\dfrac{c_1 |z_1|}{U_2^{(m+1)}(\mathbf{z})}}{1 - d_1 z_2 + \dfrac{c_1 |z_1|}{U_2^{(m+1)}(\mathbf{z})}}$$
$$\leq \frac{\kappa |z_1|}{(1 - d_1 z_2)(1 - z_2) + \kappa |z_1|},$$

for arbitraries $m \geq 3$ and $2 \leq k \leq m-1$ we get

$$\frac{c_k z_1}{U_k^{(m+1)}(\mathbf{z})U_{k+1}^{(m+1)}(\mathbf{z})} = \frac{\dfrac{c_k z_1}{U_{k+1}^{(m+1)}(\mathbf{z})}}{1 - z_2 - \dfrac{c_k z_1}{U_{k+1}^{(m+1)}(\mathbf{z})}}$$
$$\leq \frac{\dfrac{c_k |z_1|}{U_{k+1}^{(m+1)}(\mathbf{z})}}{1 - z_2 + \dfrac{c_k |z_1|}{U_{k+1}^{(m+1)}(\mathbf{z})}}$$
$$\leq \frac{\kappa |z_1|}{(1 - z_2)^2 + \kappa |z_1|},$$

and, finally, for any $m \geq 2$ we have

$$\frac{c_m z_1}{U_m^{(m+1)}(\mathbf{z})U_{m+1}^{(m+1)}(\mathbf{z})} = \frac{c_m z_1}{1 - z_2 - c_{m+1} z_1}$$
$$\leq \frac{\kappa |z_1|}{1 - z_2 + \kappa |z_1|}.$$

Thus, for $n \geq 3$ and $k \geq 2$, we obtain

$$|f_{n+k}(\mathbf{z}) - f_n(\mathbf{z})| \leq \frac{|d_0||z_2|(|z_2|(1 - z_2) + \kappa|z_1|)(1 - z_2)^{-2}(1 - z_2 + \kappa|z_1|)^{-2}(\kappa|z_1|)^{n-1}}{(1 - d_1 z_2)((1 - d_1 z_2)(1 - z_2) + \kappa|z_1|)((1 - z_2)^2 + \kappa|z_1|)^{n-3}}. \tag{2.15}$$

It is obvious that for an arbitrary fixed $\mathbf{z} \in \mathfrak{D}_\eta$

$$\frac{|d_0||z_2|(|z_2|(1 - z_2) + \kappa|z_1|)(1 - z_2)^{-2}(1 - z_2 + \kappa|z_1|)^{-2}(\kappa|z_1|)^{n-1}}{(1 - d_1 z_2)((1 - d_1 z_2)(1 - z_2) + \kappa|z_1|)((1 - z_2)^2 + \kappa|z_1|)^{n-3}} \to 0$$

as $n \to +\infty$. Therefore, the arbitrariness of k it follows (A).

Next, we will proof (B). Let $\mathfrak{L}$ denote an arbitrary compact subset of $\mathrm{Int}(\mathfrak{D}_\eta)$. Then there exists an open ball of radius L such that for $n \geq 3$, $k \geq 2$ and for all $\mathbf{z} \in \mathfrak{L}$, we get

$$|f_{n+k}(\mathbf{z}) - f_n(\mathbf{z})| < \frac{|d_0|L(L(1-\eta) + \kappa L)(1-\eta)^{-2}(1-\eta+\kappa L)^{-2}(\kappa L)^{n-1}}{(1 - d_1\eta)((1 - d_1\eta)(1-\eta) + \kappa L)((1-\eta)^2 + \kappa L)^{n-3}}$$
$$= \frac{|d_0|(1-\eta+\kappa)(1-\eta)^{-2}(1-\eta+\kappa L)^{-2}\kappa^{n-1}L^{n+1}}{(1 - d_1\eta)((1 - d_1\eta)(1-\eta) + \kappa L)((1-\eta)^2 + \kappa L)^{n-3}}.$$

Next, for arbitrary integer numbers q, p such that $q \geq 2$, $p \geq n \geq 2$, and for all $\mathbf{z} \in \mathfrak{L}$, we have

$$|f_{p+q}(\mathbf{z}) - f_p(\mathbf{z})| \leq |f_{p+q}(\mathbf{z}) - f_n(\mathbf{z})| + |f_p(\mathbf{z}) - f_n(\mathbf{z})|.$$

Furthermore, since

$$\frac{|d_0|(1-\eta+\kappa)(1-\eta)^{-2}(1-\eta+\kappa L)^{-2}\kappa^{n-1}L^{n+1}}{(1-d_1\eta)((1-d_1\eta)(1-\eta)+\kappa L)((1-\eta)^2+\kappa L)^{n-3}} \to 0$$

as $n \to +\infty$, it follows (B).

(C) follows directly from (2.15).

Finally, we will proof (D). It is obvious that

$$\frac{H_4(\alpha, \delta+1; \gamma, \delta; \mathbf{0})}{H_4(\alpha, \delta+2; \gamma, \delta+1; \mathbf{0})} = 1.$$

Then, there exists $0 < \varepsilon < 1$ such that (1.2) is an analytic function in the domain

$$\mathfrak{D}_{p,q,\varepsilon} = \left\{\mathbf{z} \in \mathbb{R}^2 : \ -p\varepsilon < z_1 < 0, \ -q\varepsilon < z_2 < 0\right\},$$

and

$$\mathfrak{D}_{p,q,\varepsilon} \subset (\mathfrak{D}_{p,q} \cap \operatorname{Int}(\mathfrak{D}_\eta)),$$

in particular,

$$\mathfrak{D}_{p,q,1/2} \subset (\mathfrak{D}_{p,q} \cap \operatorname{Int}(\mathfrak{D}_\eta)),$$

where $\mathfrak{D}_{p,q}$ is defined by (1.1).

Let $\mathbf{z}$ be an arbitrary fixed point in $\mathfrak{D}_{p,q,\varepsilon}$. It is clear that all elements of expansion (1.8) are positive numbers. This means that the approximants of (1.2) have the property of fork (see, [4, p. 29])

$$f_{2n}(\mathbf{z}) < f_{2n+2}(\mathbf{z}) < f_{2n+1}(\mathbf{z}) < f_{2n-1}(\mathbf{z}), \ n \geq 1$$

and, therefore, the sequences $\{f_{2n}(\mathbf{z})\}$ and $\{f_{2n-1}(\mathbf{z})\}$ converge to a finite value $f(\mathbf{z})$.

Let n be an arbitrary natural number. Consider the following expression

$$\frac{H_4(\alpha, \delta+1; \gamma, \delta; \mathbf{z})}{H_4(\alpha, \delta+2; \gamma, \delta+1; \mathbf{z})} - f_n(\mathbf{z}), \ n \geq 1,$$

where (see [1])

$$\frac{H_4(\alpha, \delta+1; \gamma, \delta; \mathbf{z})}{H_4(\alpha, \delta+2; \gamma, \delta+1; \mathbf{z})} = 1 + \cfrac{d_0 z_2}{1 - d_1 z_2 - \cfrac{c_1 z_1}{1 - \ddots - d_n z_2 - \cfrac{c_n z_1}{V_{n+1}^{(n+1)}(\mathbf{z})}}},$$

and

$$V_{n+1}^{(n+1)}(\mathbf{z}) = \frac{H_4(\alpha, \delta+n+2; \gamma, \delta+n+1; \mathbf{z})}{H_4(\alpha, \delta+n+3; \gamma, \delta+n+2; \mathbf{z})}.$$

Similar to (2.14), we have

$$V_k^{(n+1)}(\mathbf{z}) = 1 - d_k z_2 - \frac{c_k z_1}{V_{k+1}^{(n+1)}(\mathbf{z})}, \ 1 \leq k \leq n, \tag{2.16}$$

where

$$V_k^{(n+1)}(\mathbf{z}) = 1 - d_k z_2 - \cfrac{c_k z_1}{1 - d_{k+1} z_2 - \cfrac{c_{k+1} z_1}{1 - \ddots - d_n z_2 - \cfrac{c_n z_1}{V_{n+1}^{(n+1)}(\mathbf{z})}}}, \ 1 \leq k \leq n.$$

It is obvious that $U_k^{(n)}(\mathbf{z}) \neq 0$ and $V_k^{(n)}(\mathbf{z}) \neq 0$ for $1 \leq k \leq n$ and for $\mathbf{z} \in \mathfrak{D}_{p,q,\varepsilon}$. Using (2.13), (2.14), (2.16), and [4, Formula (3.3)], we get

$$\frac{H_4(\alpha, \delta+1; \gamma, \delta; \mathbf{z})}{H_4(\alpha, \delta+2; \gamma, \delta+1; \mathbf{z})} - f_n(\mathbf{z}) = d_0 z_1^{n-1} z_2 \left(z_2 + \frac{c_{n+1} z_1}{V_{n+1}^{(n+1)}(\mathbf{z})} \right) \prod_{r=1}^{n-1} \frac{c_r}{V_r^{(n+1)}(\mathbf{z}) U_r^{(n)}(\mathbf{z})}.$$

Then, for all $\mathbf{z} \in \mathfrak{D}_{p,q,\varepsilon}$ we have

$$f_{2n}(\mathbf{z}) < \frac{H_4(\alpha, \delta+1; \gamma, \delta; \mathbf{z})}{H_4(\alpha, \delta+2; \gamma, \delta+1; \mathbf{z})} < f_{2n-1}(\mathbf{z}).$$

Next, from the fork property of approximants of (1.2) it follows that for all $\mathbf{z} \in \mathfrak{D}_{p,q,\varepsilon}$

$$\lim_{n \to +\infty} f_{2n}(\mathbf{z}) = \lim_{n \to +\infty} f_{2n-1}(\mathbf{z}) = f(\mathbf{z})$$

and, therefore, for all $\mathbf{z} \in \mathfrak{D}_{p,q,\varepsilon}$,

$$f(\mathbf{z}) = \frac{H_4(\alpha, \delta+1; \gamma, \delta; \mathbf{z})}{H_4(\alpha, \delta+2; \gamma, \delta+1; \mathbf{z})}.$$

Finally, from [13, Theorem 3], it follows (D). □

The following result can be proved in much the same way as Theorem 2.1.

Theorem 2.2. *Let α, δ and γ be real numbers such that satisfy the inequalities*

$$\frac{\delta - \alpha}{\delta} > 0, \ 0 < b_k \leq \tau, \ k \geq 1,$$

where b_k, $k \geq 1$, are defined by (1.7), $\gamma \notin \{0, -1, -2, \ldots\}$, $\delta \notin \{0, -1, -2, \ldots\}$, τ is a positive number. Then:

(A) *The branched continued fraction (1.6) converges to a finite value $g(\mathbf{z})$ for each $\mathbf{z} \in \mathfrak{Q}_\eta$, where*

(2.17) $$\mathfrak{Q}_\eta = \left\{ \mathbf{z} \in \mathbb{R}^2 : \ z_1 \leq 0, \ z_2 \leq \eta \right\}, \ 0 < \eta < 1.$$

(B) *The convergence is uniformly on every compact subset of the domain $\operatorname{Int}(\mathfrak{Q}_\eta)$, and $g(\mathbf{z})$ is analytic function on $\operatorname{Int}(\mathfrak{Q}_\eta)$.*

(C) *For each $\mathbf{z} \in \mathfrak{Q}_\eta$ and $n \geq 2$*

$$|g(\mathbf{z}) - g_n(\mathbf{z})| \leq \frac{(|z_2|(1-z_2) + \tau|z_1|)(\tau|z_1|)^n}{(1-z_2)^3(1 - z_2 + \tau|z_1|)((1-z_2)^2 + \tau|z_1|)^{n-2}},$$

where $g_n(\mathbf{z})$ is the nth approximant of (1.6).

(D) *The function $g(\mathbf{z})$ is an analytic continuation of the function (1.3) in the domain $\operatorname{Int}(\mathfrak{Q}_\eta)$.*

By setting $\alpha = 0$ and replacing δ with $\delta - 1$ in Theorem 2.2, we have the following result.

Corollary 2.1. *Let δ and γ be a real number that satisfy condition*

$$0 < \frac{2}{\gamma} \leq \lambda, \ 0 < \frac{k(2\gamma - k - 3)}{(\gamma + k - 2)(\gamma + k - 1)} \leq \lambda, \ k \geq 2,$$

herewith $\delta \notin \{1, 0, -1, -2, \ldots\}$, $\gamma \notin \{0, -1, -2, \ldots\}$, λ is a positive number. Then:

(A) *The branched continued fraction*

(2.18) $$\cfrac{1}{1 - z_2 - \cfrac{b_1 z_1}{1 - z_2 - \cfrac{b_2 z_1}{1 - \ddots}}}$$

converges to a finite value $h(\mathbf{z})$ for each $\mathbf{z} \in \mathfrak{Q}_\eta$, where $\mathfrak{Q}_\eta$ is defined by (2.17).

(B) *The convergence is uniformly on every compact subset of the domain* $\mathrm{Int}(\mathfrak{Q}_\eta)$, *and* $h(\mathbf{z})$ *is analytic function on* $\mathrm{Int}(\mathfrak{Q}_\eta)$.
(C) *For each* $\mathbf{z} \in \mathfrak{Q}_\eta$ *and* $n \geq 3$

$$|h(\mathbf{z}) - h_n(\mathbf{z})| \leq \frac{(|z_2|(1-z_2)+\lambda|z_1|)(\lambda|z_1|)^{n-1}}{(1-z_2)^5(1-z_2+\lambda|z_1|)((1-z_2)^2+\lambda|z_1|)^{n-3}},$$

where $h_n(\mathbf{z})$ *is the* n*th approximant of (2.18).*
(D) *The function* $h(\mathbf{z})$ *is an analytic continuation of the function* $H_4(1,\delta;\gamma,\delta;\mathbf{z})$ *in the domain* $\mathrm{Int}(\mathfrak{Q}_\eta)$.

Note that in Corollary 2.1 and [8, Corollary 1] there are different conditions on the parameters δ and γ.

The truncation error bounds for branched continued fractions (1.5), (1.6) and (1.8) in $\mathbb{C}^2$ will be discussed in our next paper. The convergence of the expansions for functions (see [1])

$$\frac{H_4(\alpha,\beta;\gamma,\delta;\mathbf{z})}{H_4(\alpha+1,\beta;\gamma+1,\delta;\mathbf{z})}, \frac{H_4(\alpha,\beta;\gamma,\delta;\mathbf{z})}{H_4(\alpha+1,\beta;\gamma,\delta+1;\mathbf{z})}, \frac{H_4(\alpha,\beta;\gamma,\delta;\mathbf{z})}{H_4(\alpha,\beta+1;\gamma,\delta+1;\mathbf{z})}$$

remains open.

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