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Research Article

About Borel type relation for some positive integrals

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ABSTRACT. The manuscript contains new results describing asymptotic behavior of functions which are represented by integrals of the form $F(x) = \int_0^{+\infty} a(t)f(x+t)\nu(dt)$, where ν is locally finite measure on $\mathbb{R}_+$, a is positive ν-measurable function, f is positive and increasing to $+\infty$ in $[0,+\infty)$ function such that $f(0) = 1$ and $\ln f(x)$ is convex on the interval $[0,+\infty)$ function. The obtained main result was applied to the study of the stability of the maximum term of the series of the form $F(x) = \sum\limits_{n=0}^{\infty} a_n f(\lambda_n + x), \quad a_n \geq 0\ (n \geq 0)$.

Keywords: Maximal term, functional series, Borel relation, stability of maximal term.

2020 Mathematics Subject Classification: 30G35.

1. INTRODUCTION

Everywhere in the text below we suppose that $\lambda = (\lambda_n)$ is a positive non-decreasing sequence, i.e., $\lambda_0 = 0$ and $\lambda_n \to +\infty$ as $n \to +\infty$, and $f\colon \mathbb{R}_+ := (0,+\infty) \to \mathbb{R}_+$ is a positive Borel function. In addition, let ν be a Borel measure on the σ-algebra $\mathcal{B}(\mathbb{R}_+)$ of Borel sets from $\mathbb{R}_+$. We also denote by $I(\nu, f)$ the class of functions $F\colon \mathbb{R}_+ \to \mathbb{R}_+$ represented for each $x \in \mathbb{R}$ by the integral of the form

$$(1.1)\qquad F(x) = \int_0^{+\infty} a(t)f(tx)\nu(dt),$$

where $a\colon [0,+\infty) \to [0,+\infty)$ is a Borel function. In the case $\nu(dt) = dn_\lambda(t)$, where $n(t) = \sum_{\lambda_n \leq t} 1$ is the counting function of the sequence λ, we obtain the class $D(\lambda, f) := I(\nu, f)$ of series of the form

$$(1.2)\qquad F(x) = \sum_{n=0}^{\infty} a_n f(\lambda_n x), \quad a_n \geq 0\ (n \geq 0),$$

where $a_n = a(\lambda_n) \geq 0$ $(n \geq 0)$. For the functions $a(t)\colon [0,+\infty) \to [0,+\infty)$, $f(t)\colon [0,+\infty) \to [0,+\infty)$, a measure ν and every $x > 0$, we denote

$$\mu_*(x) = \sup\{a(t)f(tx)\colon t \in \text{supp}\ \nu\}.$$

Remark 1.1. supp $dn_\lambda = \{\lambda_n \colon n \geq 0\}$. *Therefore,*

$$\mu_*(x) = \sup\{a_n f(t\lambda_n)\colon n \geq 0\} := \mu(x, F)$$

is the maximal term of the series (1.2).

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In papers [7, 2], the conditions were obtained under which, the Borel-type asymptotic relation

$$\ln F(x) = (1 + o(1)) \ln \mu_*(x) \tag{1.3}$$

holds as $x \to +\infty$ for series of the form (1.2) outside some set of finite Lebesgue measure, where f is a positive functions on $\mathbb{R}_+$, such that the auxiliary function $y = \ln f(x) \colon \mathbb{R}_+ \to \mathbb{R}_+$ is a convex function on $\mathbb{R}_+$.

In particular, in paper [2] the following theorem was proved.

Theorem 1.1 ([2]). *If condition*

$$\int_0^{+\infty} t^{-2} \ln \nu_0(t)\, dt < +\infty \tag{1.4}$$

holds with $\nu_0(t) = \nu\{u \geq 0 : \ln f(u) \leq t\}$, then for every function $F \in I(\nu, f)$ there exists a set E of finite Lebesgue measure such that the asymptotic relation (1.3) holds as $x \to +\infty$ $(x \notin E)$.

In the case $\nu(dt) = dn_\lambda(t)$, condition (1.4) is equivalent (see results for positive integrals [6] and for the Dirichlet series [2, 4, 5]) to the condition

$$\sum_{n=1}^{+\infty} \frac{1}{n \ln f(\lambda_n)} < +\infty. \tag{1.5}$$

Therefore, from Theorem 1.1 we obtain:

Theorem 1.2 ([7]). *Let $f \colon \mathbb{R}_+ \to \mathbb{R}_+$ be a differentiable function such that the function $\ln f(x) \colon \mathbb{R}_+ \to \mathbb{R}_+$ is a convex function and a non-negative sequence (λ_n) be such that $0 \leq \lambda_n \uparrow +\infty$ $(0 \leq n \uparrow +\infty)$. If a function F represented on $(0, +\infty)$ by a series of form (1.2) and condition (1.5) holds, then there exists a set $E \subset (0, +\infty)$ of finite Lebesgue measure such that asymptotic relation (1.3) holds as $x \to +\infty$ $(x \notin E)$.*

In [6], a similar result about the Borel-type relation (1.3) was obtained for more general positive integrals of the form

$$F(x) = \int_0^{+\infty} a(t) f(tx + \beta(t)\tau(x)) \nu(dt), \tag{1.6}$$

which are, in particular, generalizations of the Taylor-Dirichlet type series. Here $a(t)$, $\beta(t)$ are the positive Borel functions and a function f is the same as above.

Theorem 1.3 ([6]). *Let F be a function of form (1.6). If function $\tau(x) \colon \mathbb{R}_+ \to \mathbb{R}_+$ is differentiable on $[x_0, +\infty)$ such that $\tau'(x) \geq 1$ $(x \geq x_0)$ and the condition (1.4) holds with*

$$\nu_0(t) = \nu\big(\{u \geq 0 \colon \ \ln f(u + \beta(u)) \leq t\}\big),$$

then for every function F of form (1.6) there exists a set E of finite Lebesgue measure such that the asymptotic relation (1.3) holds as $x \to +\infty$ $(x \notin E)$.

In article [8] (see also [14, 15, 16]), there was considered regularly convergent series of the form

$$G(z) = \sum_{n=0}^{+\infty} b_n g(z\beta_n), \tag{1.7}$$

that is $\mathfrak{M}_G(r) := \sum_{n=0}^{+\infty} |b_n| M_g(r\beta_n) < +\infty$ for all $r > 0$, where $g(z)$ is some entire function, (β_n) is a given non-negative sequence such that $\beta_n \uparrow +\infty$ $(n \uparrow +\infty)$ and $M_g(r) = \max\{|g(z)| \colon |z| = r\}$. If we now denote $F(x) = \mathfrak{M}_G(x)$, $f(x) = M_g(x)$, $\lambda_n = \beta_n$, then we obtain a series of form

(1.2), where the function $\ln f(x)$ is logarithmically convex, that is, the function $h(x) := \ln f(e^x)$ is a convex function on $\mathbb{R}$. The growth and expansion properties of similar series were also studied in [9, 10, 17] the Hadamard composition of such series was considered in [11], the boundedness of the $\ell - \mathfrak{M}$-index for them recently was examined in [12, 13].

For the function $f(x) = e^x$ we obtain an entire Dirichlet series $F(x) = \sum_{n=1}^{\infty} a_n e^{x\lambda_n}$. Then, provided $\sum_{n=1}^{+\infty} 1/\big(n \ln f(\lambda_n)\big) = \sum_{n=1}^{+\infty} 1/\big(n\lambda_n\big) < +\infty$, from Theorem 1.2 it follows the statement of theorem in [4] for entire Dirichlet series. Note that in the case where the function $\ln f(x)$ is not convex, we cannot apply the statement of Theorem 1.2. The following conjecture from the article [8] is particularly relevant to this circumstance.

Let us denote $\Gamma_f(x) = x(\ln f(x))'_+$.

Conjecture 1.1 ([8]). *If*

$$\sum_{n=1}^{\infty} \frac{1}{n\Gamma_f(\lambda_n)} < +\infty, \tag{1.8}$$

then asymptotic relation (1.3) *holds as* $x \to +\infty$ *outside some exceptional set* E *such that* $\int_E \Gamma_f(x)\frac{dx}{x} < +\infty$ *for every functions* F *of form* (1.2).

For each entire transcendental function f we get $\Gamma_f(r) = r(\ln M_f(r))'_+ \nearrow +\infty$ $(r \to +\infty)$. Therefore, from condition $\int_E \Gamma_f(x)\frac{dx}{x} < +\infty$ it follows that $\int_E \frac{dx}{x} < +\infty$, that is, a set E has finite logarithmic measure (it was incorrectly written in [3] that the set E has finite Lebesgue measure).

Note that in different cases the condition (1.5) can be either weaker than the condition (1.8) or stronger. Indeed, if we choose ([3]) $\ln f(t) = (\ln t)^{1+\varrho}$, $\varrho > 0$, then $\Gamma_f(r) = (1+\varrho)(\ln t)^{\varrho}$ and the condition (1.5) is weaker than the condition (1.8). However, in the case of $\Gamma_f(r)/r \nearrow +\infty$ $(r_0 \le r \to +\infty)$, we have

$$\ln f(x) - \ln f(r_0) = \int_{r_0}^{r} \frac{\Gamma_f(t)}{t} dt \le \Gamma_f(r) \quad (r \to +\infty).$$

Therefore, the condition (1.5) is, in general, stronger than the condition (1.8). However, there is a nuance. Under the conditions of Sheremeta's conjecture the function $\ln f(x)$ should be considered logarithmically convex. And in the statement of Theorem 1.2, the function $\ln f(x)$ is convex. One should observe that the condition $\Gamma_f(r)/r \nearrow +\infty$ $(r_0 \le r \to +\infty)$ means that as in Theorem 1.2 the function $\ln f(x)$ is convex. Let us now formulate the following conjecture.

Conjecture 1.2. *The statement of Conjecture 1.1* ([8]) *holds in case if a function* f *is such that the function* $\ln f(x)$ *is convex.*

Similar conjecture we formulate also about the Borel relation for the class $I(\nu, f)$.

Conjecture 1.3. *If condition* (1.4) *holds with* $\nu_0(t) = \nu\{u \ge 0 : \ln f(u) \le t\}$ *and a function* f *is such that the function* $\ln f(x)$ *is convex, then for every function* $F \in I(\nu, f)$ *there exists a set* E *such that* $\int_{x_0}^{+\infty} \frac{\Gamma_f(x)}{x} dx < +\infty$ *and the asymptotic relation* (1.3) *holds as* $x \to +\infty$ $(x \notin E)$.

2. Main Result

Let $I_+(\nu, f)$ be the class of functions $F : \mathbb{R}_+ \to \mathbb{R}_+$ represented by integrals of the form

$$F(x) = \int_0^{+\infty} a(t) f(x+t) \nu(dt),$$

where ν is locally finite measure on $\mathbb{R}_+$, a is positive ν-measurable function, f is positive and increasing to $+\infty$ in $[0,+\infty)$ function such that $f(0)=1$ and $\ln f(x)$ is convex on the interval $[0,+\infty)$ function.

The following statement is an analogue of Theorem 3.1 for integrals from the class $I_+(\nu,f)$ and gives a positive answer to Conjecture 1 from [1].

Proposition 2.1. *If condition*

$$\int_0^{+\infty} t^{-2}\ln\nu_0(t)\,dt<+\infty \tag{2.9}$$

holds with $\nu_0(t)=\nu\{u\ge 0:(\ln f(u))'\le t\}$, then for every function $F\in I_+(\nu,f)$ there exists a set E of finite Lebesgue measure such that the asymptotic relation

$$\ln F(x)\le(1+o(1))\ln\mu(x,F)$$

holds as $x\to+\infty$, $(x\notin E)$, where $\mu(x,F)=\sup\{a(t)f(x+t):x\in \mathrm{supp}\,\nu\}$ and $\mathrm{supp}\,\nu$ is the support of the measure ν.

Proof. Denote $g(x)=\ln F(x)$. In the following proof, we reason similarly to the proof of Corollary 3.1 in [3]. We assume that $f'(x)$ denotes the right-hand derivative.

Since $g_0(x):=\ln f(x)$ is convex function on $\mathbb{R}_+$, $(\ln f(t))'\le(\ln f(u))'\big|_{u=x+t}$ and $f'(x+t)>0$ for fixed $x>0$ and for every $t>0$. Then for fixed $x>0$

$$G:=G_x=\Big\{t>0\colon(\ln f(u))'\Big|_{u=x+t}\le 2g'(x)\Big\}\subset\{t>0\colon(\ln f(t))'\le 2g'(x)\}:=G_0,$$

where $g(x):=\ln F(x)$. Hence,

$$\nu(G)\le\nu(G_0)=\nu_0(2g'(x)).$$

For $x>0$ and $t\notin G_x$, we have $(\ln f(u))'\Big|_{u=x+t}>2g'(x)$, i.e.,

$$\begin{aligned}\int\limits_{\mathbb{R}_+\setminus G}a(t)f(t+x)\nu(dt)&=\int\limits_{\mathbb{R}_+\setminus G}a(t)\frac{f'(t+x)}{(\ln f(u))'\big|_{u=t+x}}\nu(dt)\\&\le\frac{1}{2g'(x)}\int\limits_{\mathbb{R}_+\setminus G}a(t)f'(t+x)\nu(dt)\\&\le\frac{F(x)}{2F'(x)}\int\limits_{\mathbb{R}_+}a(t)f'(t+x)\nu(dt)=\frac{F(x)}{2},\end{aligned}$$

because $F'(x)=\int\limits_{\mathbb{R}_+}a(t)f'(t+x)\nu(dt)$. So,

$$\begin{aligned}F(x)&=\int\limits_G a(t)f(t+x)\nu(dt)+\int_{\mathbb{R}_+\setminus G}a(t)f(t+x)\nu(dt)\\&\le\int\limits_G a(t)f(t+x)\nu(dt)+\frac{F(x)}{2}.\end{aligned}$$

Therefore,

$$F(x)\le 2\int\limits_G a(t)f(t+x)\nu(dt)\le 2\mu(x,F)\nu(G)\le\mu(x,F)\nu_0(2g'(x)),\ x\ge x_0, \tag{2.10}$$

where $\nu_0(t) := \nu\{u \geq 0\colon (\ln f(u))' \leq t\}$.

Now we need the following lemma ([6, 5]).

Lemma 2.1. *For a given non-decreasing function $\nu_0(t)\colon \mathbb{R}_+ \to \mathbb{R}_+$ the condition*

$$(\exists t_0 > 0)\colon \quad \int_{t_0}^{+\infty} \frac{d\ln\nu_0(t)}{t} < +\infty, \tag{2.11}$$

is equivalent to each of the following two conditions: i) condition (2.9); ii) there exists a continuous function $\psi\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that $\psi(t) \uparrow +\infty$ $(t_0 \leq t \to +\infty)$ and

$$\int_0^{+\infty} \frac{dt}{\psi(t)} < +\infty, \qquad \ln\nu_0(t) = o(\psi^{-1}(t)) \quad (t \to +\infty). \tag{2.12}$$

For the function $\psi_0(t) = \psi(t)/2$ let us denote a set

$$E := \{x > x_0\colon g'(x) \geq \psi_0(g(x))\}.$$

Then, we obtain the following estimate of the Lebesgue measure for the set E

$$\text{meas}\,(E \cap [x_0, +\infty)) = \int_E dx \leq \int_E \frac{g'(x)}{\psi_0(g(x))}dx \leq \int_0^{+\infty} \frac{dt}{\psi_0(t)} < +\infty, \tag{2.13}$$

i.e., the set E has finite Lebesgue measure. Therefore, from inequality (2.10) and relation (2.12) we obtain finally

$$\begin{aligned}\ln F(x) &\leq \ln 2 + \ln\mu(x, F) + \ln\nu_0(2g'(x)) \\ &\leq \ln 2 + \ln\mu(x, F) + \ln\nu_0(\psi(g(x))) = \ln\mu(x, F) + o(g(x))\end{aligned}$$

as $x \to +\infty$ $(x \notin E)$. Hence, $\ln F(x) \leq (1 + o(1))\ln\mu(x, F)$ as $x \to +\infty$ $(x \notin E)$.

□

3. Corollary: Stability of a maximal term

In view of Proposition 2.1 and Lemma 2.1, we obtain the following corollaries.

Corollary 3.1 ([3], Theorem 2)**.** *Let $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a differentiable function such that the function $y = \ln f(x)\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a convex function and the right-hand derivative $L(x, f) := (\ln f(x))'_+ \uparrow +\infty$ $(x \geq x_0)$; (λ_n) be a non-negative sequence such that $0 \leq \lambda_n \uparrow +\infty$ $(0 \leq n \uparrow +\infty)$. If a function F represented on $(0, +\infty)$ by a series of form*

$$F(x) = \sum_{n=0}^{\infty} a_n f(\lambda_n + x), \quad a_n \geq 0 \quad (n \geq 0), \tag{3.14}$$

and the condition

$$\sum_{n=1}^{+\infty} \frac{1}{nL(\lambda_n, f)} < +\infty \tag{3.15}$$

holds, then there exists a set $E \subset (0, +\infty)$ of finite Lebesgue measure such that asymptotic relation $\ln F(x) = (1 + o(1))\ln\mu_F(x)$ holds as $x \to +\infty$ $(x \to +\infty,\ x \notin E)$, where

$$\mu_F(x) = \max\{a_n f(\lambda_n + x)\colon n \geq 0\}.$$

Hence, for series of the form (1.7) we obtain the following corollary.

Corollary 3.2 ([3, Corollary 1])**.** *Let (β_n) be a non-decreasing to $+\infty$ sequence and a function G represented by regularly convergent functional series of form (1.7), where g is an entire function. If condition (1.8) satisfies, then the relation $\ln M_G(r) = (1 + o(1))\ln\mu_G(r)$ holds as $r \to +\infty$ outside a set of finite logarithmic measure, where $\mu_G(r) = \max\{|b_n|M_g(r\beta_n)\colon n \geq 0\}$.*

Let L be a class of positive continuous on $\mathbb{R}_+ := [0, +\infty)$ functions $l(t)$ such that $l(t) \to +\infty$ $(t \to +\infty)$. By L_+ we denote the subclass of L such that $l(t) \uparrow +\infty$ as $t \to +\infty$, and by $\mathcal{W}$ the class of functions $w \in L_+$ such that

$$\int_1^{+\infty} x^{-2} w(x)dx < +\infty.$$

Let us denote by $D_+(\lambda, f)$ the class of functions $F\colon \mathbb{R}_+ \to \mathbb{R}_+$ of form (3.14). For a series $F \in D_+(\lambda, f)$ and any sequence (b_n), $b_n \in \mathbb{R}_+ \setminus \{0\}$ $(n \geq 0)$ we consider

$$B^+(x) = \sum_{n=0}^{+\infty} a_n b_n f(x + \lambda_n), \qquad B^-(x) = \sum_{n=0}^{+\infty} a_n b_n^{-1} f(x + \lambda_n).$$

We call that a series of the form (3.14) (maximal term of the series) is stable if the relations

$$\ln \mu(x, F) = (1 + o(1)) \ln \mu(x, B^+) = (1 + o(1)) \ln \mu(x, B^-) \tag{3.16}$$

hold as $x \to +\infty$ outside some set $E \subset [0, +\infty)$ of the finite Lebesgue measure, i.e., $\mathrm{meas}\, E := \int_E dx < +\infty$.

For a function $w \in L$ let us denote

$$B_w(x) = \sum_{n=0}^{+\infty} a_n e^{w(\lambda_n)} f(x\lambda_n).$$

Let us denote

$$\nu_0(t) = \nu\{u \geq 0\colon (\ln f(u))' \leq t\}, \quad \nu(G) = \sum\nolimits_{\lambda_n \in G} e^{w(\lambda_n)}$$

for every bounded set $G \in \mathbb{R}_+$.

Theorem 3.4. *Let $F \in D_+(\lambda, f)$. If there exists a function $w \in L_+$ such that $B_w \in D_+(\lambda, f)$, $\ln \nu_0 \in \mathcal{W}$ and inequalities*

$$e^{-w(\lambda_n)} \leq b_n \leq e^{w(\lambda_n)} \quad (n \geq k_1), \tag{3.17}$$

are valid, then there exists a set $E \subset \mathbb{R}_+$ of finite Lebesque measure such that relation

$$\ln \mu(x, F) = (1 + o(1)) \ln \mu(x, B_+) = (1 + o(1)) \ln \mu(x, B_-) \tag{3.18}$$

holds as $x \to +\infty$ $(x \notin E)$.

Proof of Theorem 3.4. Note that relation (3.18) will follow from the fact that

$$\ln \mu(x, F) = (1 + o(1)) \ln \mu(x, B_w) \tag{3.19}$$

as $x \to +\infty$ outside of some set E of finite Lebesgue measure. Let us prove relation (3.19). Let $a(t), b(t)$ be measurable nonnegative functions on $\mathbb{R}_+$ such that $a(\lambda_n) = a_n$, $b(\lambda_n) = e^{w(\lambda_n)}$ and

$$\mu(x, F) = \sup\{a(t)f(t + x) : \ t \in \mathbb{R}_+\}, \ \ \mu(x, B_w) = \sup\{a(t)b(t)f(t + x) : \ t \in \mathbb{R}_+\}.$$

It is enough to take that $a(t) = 0$ for $t \notin \{\lambda_n : n \in \mathbb{Z}_+\}$. Then for all $x \in \mathbb{R}$ we get

$$\mu(x, F) \leq \mu(x, B_w) \leq B_w(x) = \sum_{n=0}^{+\infty} a_n b(\lambda_n) f(x + \lambda_n) = \int_{\mathbb{R}_+} a(t)f(t + x)\nu(dt), \tag{3.20}$$

where measure ν is such that $\nu(G) = \sum_{n=0}^{+\infty} b(\lambda_n)\delta_{\lambda_n}(G)$ for each bounded set $G \subset \mathbb{R}_+$ and $\delta_\lambda(G) = 1$ for $\lambda \in G$ and $\delta_\lambda(G) = 0$ for $\lambda \notin G$. Since G is bounded and $\lambda_n \to +\infty$, it means that the finite number of λ_n belongs to the set G, i.e., for all $n \geq n_0(G)$ one has $\delta_{\lambda_n}(G) = 0$. In view of this, the series $\sum_{n=0}^{+\infty} b(\lambda_n)\delta_{\lambda_n}(G)$ reduces to the finite sum $\sum_{n=0}^{n_0} b(\lambda_n)\delta_{\lambda_n}(G)$. It yields σ-additivity of the measure ν.

From the condition $\ln\nu_0 \in \mathcal{W}$ we immediately get that condition (2.9) of Proposition 2.1 is satisfied, because we substitute $w(x) = \ln\nu_0(x)$ in (2.9). Applying Proposition 2.1 to the integral in (3.20), as $x \to +\infty$ $(x \notin E)$, (here a set E is such as in Proposition 2.1) we obtain

$$\ln\mu(x, F) \leq \ln\mu(x, B_w) \leq (1 + o(1))\ln\mu_*(x),$$

where $\mu_*(x) = \max\{a(t)f(x + t)\colon t \in \mathbb{R}_+\}$. As for the choice of function $a(t)$, we get $\mu_*(x) = \mu(x, F)$ and deduce relation (3.19). □

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