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Research Article

Maximum modulus of slice entire regular functions of quaternionic variable with bounded index

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ABSTRACT. The manuscript contains new results describing local behavior of slice entire regular functions of quaternionic variable. There is selected such their subclass as functions having bounded index and for these functions we describe uniform estimate of their maximum modulus in a disc of larger radius by their maximum modulus in a disc of lesser radius multiplied on some constant.

Keywords: Slice entire function, bounded index, maximum modulus, local behavior.

2020 Mathematics Subject Classification: 30G35.

1. INTRODUCTION

The paper is a continuation of investigations initiated in [1, 2]. Among them, paper [2] was an attempt to introduce a notion of bounded index for Fueter regular functions of quaternionic variable (see more details on the notion of regularity in [14, 19, 20]). The approach considers all possible partial derivatives in real components of the quaternionic variable and it is close to bounded index in joint variables [3, 6, 9]. It can also be applicable to other non-commutative algebras, for example, such as in [19, 18]. In the second paper [1], there was studied a similar problem for another concept of regularity – so-called slice regularity [15, 16, 17]. This notion is very flexible in the quaternionic analysis because it allows to deduce more quaternionic analogs of known statements from the complex analysis [13].

One should observe that the notion of slice holomorphy exists even in the multidimensional complex analysis [4, 8, 12]. However, in this context, a slice holomorphic function of several complex variables is understood to be a function that is holomorphic on all slices in one fixed direction, and in order to cover the whole n-dimensional complex space [7, 21] (either a unit ball [11], or unit polydisc [10]), we change the initial point of this slice in the form $z^0 + t\mathbf{b}$, where $z^0 \in \mathbb{C}^n$ is the start point, $\mathbf{b} \in \mathbb{C}^n$ is the fixed direction, and t is the variable parameter. In contrast, the quaternionic analysis considers slices of the form $x + Iy$, with $x, y \in \mathbb{R}$ and I is the variable point from the unit sphere of purely imaginary quaternions. In other words, the direction changes here, but for every fixed I it is possible to use classical results from the complex analysis.

Our goal is to combine the notion of slice regularity with the notion of bounded index and to deduce new results. It is important in view of known applications of functions having bounded index to analytic theory of differential equations [11, 12, 23].

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2. Main definitions and notations

We will use standard notations of quaternionic analysis from [1, 13]. Let $\mathbb{H}$ be the skew field of quaternions which is defined as $\mathbb{H} = \{g = x_0 + ix_1 + jx_2 + kx_3 \colon x_0, x_1, x_2, x_3 \in \mathbb{R}\}$, where the imaginary units i, j, k satisfy $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$. It is a non-commutative field. We define the Euclidean norm on $\mathbb{H} : |q| = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}$. The quantities $Re\, q = x_0$, and $Im\, q = ix_1 + jx_2 + kx_3$, we call the real and imaginary parts of the quaternion $q = x_0 + ix_1 + jx_2 + kx_3 \in \mathbb{H}$, respectively. The symbol $\mathbb{S}$ denotes the unit sphere (it is a sphere in $\mathbb{R}^3$ and it is a cylinder in $\mathbb{H}$, i.e., in $\mathbb{R}^4$) of purely imaginary quaternions, i.e., $\mathbb{S} = \{q = ix_1 + jx_2 + kx_3 \colon x_1^2 + x_2^2 + x_3^2 = 1\}$. One should observe that if $I \in \mathbb{S}$, then $I^2 = -1$. Given this, the elements of $\mathbb{S}$ are also called imaginary units. For any fixed $I \in \mathbb{S}$ we define the 'complex plane' $\mathbb{C}_I := \{x + Iy : x, y \in \mathbb{R}\}$. It is easy to check that $\mathbb{C}_I$ is isomorphic with the complex plane $\mathbb{C}$. Moreover, $\mathbb{H} = \bigcup_{I\in\mathbb{S}} \mathbb{C}_I$. In this case, the real axis $\{q\colon\ Im\, q = 0\}$ belongs to $\mathbb{C}_I$ for every $I \in \mathbb{S}$ and thus a real quaternion can be associated with any imaginary unit I. Any non-real quaternion $q = x_0 + ix_1 + jx_2 + kx_3$ is uniquely associated to the element $I_q \in \mathbb{S}$ defined by

$$I_q := \frac{ix_1 + jx_2 + kx_3}{|ix_1 + jx_2 + kx_3|}.$$

It is obvious that q belongs to the complex plane $\mathbb{C}_{I_q}$.

Let $f : \mathbb{H} \to \mathbb{H}$ be real differentiable. The function f is said to be (left) entire slice regular, if for every $I \in \mathbb{S}$ its restriction f_I to the complex plane $\mathbb{C}_I = \mathbb{R} + I\mathbb{R}$ passing through origin and containing I and 1 satisfies $\bar{\partial}_I f(x + Iy) := \frac{1}{2}\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right) f_I(x + Iy) = 0$ on $\mathbb{C}_I$. The class of (left) slice regular functions on $\mathbb{H}$ will be denoted by $\mathcal{R}(\mathbb{H})$. Analogously, a function f is said to be right entire slice regular in $\mathbb{H}$ if $(f\bar{\partial}_I)(x + Iy) := \frac{1}{2}\left(\frac{\partial}{\partial x} f_I(x + Iy) + \frac{\partial}{\partial y} f_I(x + Iy)I\right) = 0$ on $\mathbb{C}_I$. Let $f \in \mathcal{R}(\mathbb{H})$. The so-called left I-derivative of f at a point $q = x + Iy$ is defined by $\partial_I f_I(x+Iy) := \frac{1}{2}\left(\frac{\partial}{\partial x} f_I(x + Iy) - I\frac{\partial}{\partial y} f_I(x + Iy)\right)$ and the right I-derivative of f at $q = x + Iy$ is defined by $\partial_I f_I(x+Iy) := \frac{1}{2}\left(\frac{\partial}{\partial x} f_I(x + Iy) - \frac{\partial}{\partial y} f_I(x + Iy)I\right)$. Let us now introduce another suitable notion of derivative. The slice derivative $\partial_s f$ of f, is defined by:

$$\partial_s(f)(q) = \begin{cases} \partial_I(f)(q), & \text{if } q = x + Iy,\ y \neq 0, \\ \frac{\partial f}{\partial x}(x), & \text{if } q = x \in \mathbb{R}. \end{cases}$$

We will often write $f'(q)$ instead of $\partial_s f(q)$. The k-th derivative of $f \in \mathcal{R}(\mathbb{H})$ is defined recursively as $f^{(k)}(q) = (f^{(k-1)}(q))'$. It is important to note that if $f(q)$ is a slice regular function then also $f'(q)$ is a slice regular function.

A function $f \in \mathcal{R}(\mathbb{H})$ is called a function of bounded index (see [1]), if there exists $m_0 \in \mathbb{Z}_+$ such that for every $m \in \mathbb{Z}_+$ and every $q \in \mathbb{H}$ the following inequality is valid

$$\frac{|f^{(m)}(q)|}{m!} \leq \max\left\{\frac{|f^{(k)}(q)|}{k!} : 0 \leq k \leq m_0\right\}. \tag{2.1}$$

The least such integer m_0 is called the index of the entire slice regular function f and is denoted by $N(f) = m_0$. As an addendum to notion of bounded index there is known a notion of bounded $\mathfrak{M}$-index [22] which allows to examine analytic functions with unbounded multiplicities of zeros.

There were obtained two following criteria of index boundedness for slice entire regular functions in [1].

Theorem 2.1 ([1]). *A function $f \in \mathcal{R}(\mathbb{H})$ is of bounded index if and only if for every $\eta > 0$ there exist $n_0 = n_0(\eta) \in \mathbb{Z}_+$ and $P_1 = P_1(\eta) \geq 1$ such that for any $I \in \mathbb{S}$ and for every $x_0 \in \mathbb{R}$, $y_0 \in \mathbb{R}$ (or, equivalently, for any $q = x_0 + Iy_0 \in \mathbb{H}$) there exists $k_0 = k_0(q) = k_0(x_0, y_0, I) \in \mathbb{Z}_+$ with $0 \leq k_0 \leq n_0$ and the following inequality holds*

$$\max\{|f_I^{(k_0)}(x+Iy)| : \sqrt{(x-x_0)^2+(y-y_0)^2} \leq \eta\} \leq P_1|f_I^{(k_0)}(x_0+Iy_0)|. \tag{2.2}$$

Theorem 2.2 ([1]). *Let $f \in \mathcal{R}(\mathbb{H})$. If there exist $\eta > 0$, $n_0 = n_0(\eta) \in \mathbb{Z}_+$ and $P_1 = P_1(\eta) \geq 1$ such that for any $I \in \mathbb{S}$ and for every $x_0 \in \mathbb{R}$, $y_0 \in \mathbb{R}$ there exists $k_0 = k_0(q) = k_0(x_0, y_0, I) \in \mathbb{Z}_+$, with $0 \leq k_0 \leq n_0$, and (2.2) is satisfied, then the function f has bounded index.*

3. Estimate of the maximum modulus on a larger circle by the maximum modulus on a smaller circle.

Now we will try to estimate the maximum modulus of entire slice regular function on a circle. Using Theorem 2.1, we prove a criterion of index boundedness in direction. This result was announced under a talk at the International Workshop on Modern Problems of Analysis, Optimization, Approximation and Their Applications [5].

Theorem 3.3. *A function $f \in \mathcal{R}(\mathbb{H})$ is of bounded index if and only if for each pair of radii r_1 and r_2 with $0 < r_1 < r_2 < \infty$ there exists number $P_1 \geq 1$ depending only on r_1 and r_2, i.e., $P_1 = P_1(r_1, r_2)$, such that for any $I \in \mathbb{S}$ and for every $x_0 \in \mathbb{R}$, $y_0 \in \mathbb{R}$ (or, equivalently, for any $q = x_0 + Iy_0 \in \mathbb{H}$) the following inequality holds*

$$\begin{aligned} &\max\{|f_I(x+Iy)| : \sqrt{(x-x_0)^2+(y-y_0)^2} = r_1\} \\ \leq &P_1 \max\{|f_I(x+Iy)| : \sqrt{(x-x_0)^2+(y-y_0)^2} = r_2\}, \end{aligned} \tag{3.3}$$

where maxima in equation (3.3) are taken over circles within the corresponding complex slice.

Proof. **Necessity.** Let f be of bounded index and $N(f) \equiv N < +\infty$. We will prove the converse statement and introduce a proof by contradiction. Suppose that there exist radii r_1 and r_2, $0 < r_1 < r_2 < \infty$, such that for any constant $P \geq 1$ there exist an imaginary unit $I_* \in \mathbb{S}$ and appropriate $x_* \in \mathbb{R}$, $y_* \in \mathbb{R}$ (or, equivalently, some quaternionic point $q_* = x_* + I_* y_* \in \mathbb{H}$) providing validity of the following inequality

$$\begin{aligned} &\max\{|f_{I_*}(x+I_*y)| : \sqrt{(x-x_*)^2+(y-y_*)^2} = r_2\} \\ > &P \max\{|f_{I_*}(x+I_*y)| : \sqrt{(x-x_*)^2+(y-y_*)^2} = r_1\}. \end{aligned} \tag{3.4}$$

In Theorem 2.1, we put $\eta = r_2$ and apply this theorem. Then there exist $n_0 = n_0(r_2) \in \mathbb{Z}_+$ and $P_1 = P_1(r_2) \geq 1$ such that for every $I \in \mathbb{S}$ and for every $x_0 \in \mathbb{R}$, $y_0 \in \mathbb{R}_+$ and some $k_0 = k_0(q) = k_0(x_0, y_0, I) \in \mathbb{Z}_+$, $0 \leq k_0 \leq n_0$, one has

$$\max\{|f_I^{(k_0)}(x+Iy)| : \sqrt{(x-x_0)^2+(y-y_0)^2} \leq r_2\} \leq P_1|f_I^{(k_0)}(x_0+Iy_0)|. \tag{3.5}$$

In view of the arbitrariness of $I \in \mathbb{S}$, $x_0 \in \mathbb{R}$, $y_0 \in \mathbb{R}_+$, in (3.5) we substitute $I = I_*$, $x_0 = x_*$, $y_0 = y_*$ from (3.4). The choice of I does not affect the index $N(f)$. Then for some k_0 it yields

$$\max\{|f_{I_*}^{(k_0)}(x+I_*y)| : \sqrt{(x-x_*)^2+(y-y_*)^2} \leq r_2\} \leq P_1|f_{I_*}^{(k_0)}(x_*+I_*y_*)|. \tag{3.6}$$

For further proof, we need the Maximum Modulus Principle for slice entire regular functions. Let $U \subset \mathbb{H}$. By Definition 2.4 from [13] the domain U is a slice domain if it is a connected set, whose intersection with every complex plane $\mathbb{C}_I$ is connected. By the Maximum Modulus Principle for slice domains (see Theorem 3.12 in [13, P. 45]), if U is a slice domain, $f : U \to \mathbb{H}$

is slice regular and $|f|$ has a relative maximum at $p \in U$, then f is constant. The principle is applicable in our case because for given $I \in \mathbb{S}$ the domain

$$\left\{x + Iy\colon \sqrt{(x-x_0)^2 + (y-y_0)^2} \le \frac{p\eta}{g(\eta)}\right\}$$

is a slice domain. The Maximum Modulus Principle holds for each slice $\mathbb{C}_I$, not for the entire quaternionic space $\mathbb{H}$.

First, we consider the case $k_0 = 0$. Then (3.6) and double application of the Maximum Modulus Principle yields

$$\begin{aligned}
&\max\left\{|f_{I_*}(x+I_*y)| : \sqrt{(x-x_*)^2+(y-y_*)^2} = r_2\right\}\\
=&\max\left\{|f_{I_*}(x+I_*y)| : \sqrt{(x-x_*)^2+(y-y_*)^2} \le r_2\right\}\\
\le& P_1|f_{I_*}(x_*+I_*y_*)|\\
\le& P_1\max\left\{|f_{I_*}(x+I_*y)| : \sqrt{(x-x_*)^2+(y-y_*)^2} \le r_1\right\}\\
=& P_1\max\left\{|f_{I_*}(x+I_*y)| : \sqrt{(x-x_*)^2+(y-y_*)^2} = r_1\right\}.
\end{aligned}$$

Hence, estimate (3.3) holds and the necessity is proved in the case.

Suppose that $k_0 > 0$. Put in (3.4)

$$P = n_0!\left(\frac{r_2}{r_1}\right)^{n_0}\left(P_1 + \frac{r_1}{r_2 - r_1}\right) + 1. \tag{3.7}$$

Let us introduce auxiliary maximum modulus points on the slice discs. We assume $\mathring{q} = \mathring{x} + I_*\mathring{y} \in H$ is such that

$$|f_I(\mathring{x} + I_*\mathring{y})| = \max\left\{|f_{I_*}(x+I_*y)| : \sqrt{(x-x_*)^2+(y-y_*)^2} = r_1\right\},$$

and $q_{*j} = x_{*j} + I_*y_{*j} \in \mathbb{H}$ is such that

$$|f^{(j)}_{I_*}(x_{*j} + I_*y_{*j})| = \max\{|f^{(j)}_{I_*}(x+I_*y)| : \sqrt{(x-x_*)^2+(y-y_*)^2} = r_2\},$$

$j \in \mathbb{Z}_+$.

In the case $|f_{I_*}(\mathring{x}+I_*\mathring{y})| = 0$ it follows that for all $x \in \mathbb{R}$ and $y \in \mathbb{R}$ with $(x-x_*)^2+(y-y_*)^2 = r_1^2$ one has $f_{I_*}(\mathring{x} + I_*\mathring{y}) \equiv 0$. Further let us remind Identity Principle for slice entire regular functions (see [13, Theorem 2.3, p. 11]):

Let $f : U \to \mathbb{H}$ be a slice regular function on a slice domain U, $Z_f = \{q \in U : f(q) = 0\}$ be the zero set of f. If there exists $I \in \mathbb{S}$ such that $\mathbb{C}_I \cap Z_f$ has an accumulation point, then $f \equiv 0$ on U.

By the Identity Principle, we obtain $f(q) \equiv 0$ which contradicts (3.4).

So, let $|f_{I_*}(\mathring{x} + I_*\mathring{y})| > 0$. We will use the following Cauchy estimates for slice entire regular functions. Let us cite Proposition 3.1 in [13, P. 32]. Let $F : U \to \mathbb{H}$ be a slice regular function and let $q \in U \cap \mathbb{C}_I$. For all discs $B_I(q,R) = B(q,R) \cap \mathbb{C}_I$, $R > 0$ such that $B_I(q,R) \subset U \cap \mathbb{C}_I$ the following formula holds:

$$|F^{(n)}(q)| \le \frac{n!}{R^n}\max_{s\in\partial B_I(q,R)}|F(s)|. \tag{3.8}$$

By Cauchy's inequality for $n = j$, $R = r_1$ and $q = x_* + I_* y_*$ from (3.8) we obtain

$$\frac{|f^{(j)}_{I_*}(x_* + I_* y_*)|}{j!} \le \left(\frac{1}{r_1}\right)^j \max\left\{|f_{I_*}(x + I_* y)| : \sqrt{(x - x_*)^2 + (y - y_*)^2} = r_1\right\}$$
$$= \left(\frac{1}{r_1}\right)^j |f_{I_*}(\mathring{x} + I_* \mathring{y})|, \quad j \in \mathbb{Z}_+, \tag{3.9}$$

and

$$\left|f^{(j)}_{I_*}(x_{*j} + I_* y_{*j}) - f^{(j)}_{I_*}(x_* + I_* y_*)\right| = \left|\int_{x_* + I_* y_*}^{x_{*j} + I_* y_{*j}} f^{(j+1)}_{I_*}(t)\, dt\right|$$
$$\le \left|f_{I_*}(x_{*(j+1)} + I_* y_{*(j+1)})\right| r_2. \tag{3.10}$$

From (3.9) and (3.10) we have

$$|f^{(j+1)}_{I_*}(x_{*(j+1)} + I_* y_{*(j+1)})| \ge \frac{1}{r_2}\left\{f^{(j)}_{I_*}(x_{*j} + I_* y_{*j})| - |f^{(j)}_{I_*}(x_* + I_* y_*)|\right\}$$
$$\ge \frac{1}{r_2}\left|f^{(j)}_{I_*}(x_{*j} + I_* y_{*j})\right| - \frac{j!}{r_2 (r_1)^j}|f_{I_*}(\mathring{x} + I_* \mathring{y})|,$$

where $j \in \mathbb{Z}_+$. Hence, for $k_0 \ge 1$ we get

$$|f^{(k_0)}_{I_*}(x_{*k_0} + I_* y_{*k_0})| \ge \frac{1}{r_2}|f^{(k_0-1)}_{I_*}(x_{*(k_0-1)} + I_* y_{*(k_0-1)})| - \frac{(k_0-1)!}{r_2 (r_1)^{k_0-1}}|f_{I_*}(\mathring{x} + I_* \mathring{y})|$$
$$\ge \ldots \ge \frac{1}{(r_2)^{k_0}}|f_{I_*}(x_{*0} + I_* y_{*0})|$$
$$- \left(\frac{0!}{(r_2)^{k_0}} + \frac{1!}{(r_2)^{k_0-1} r_1} + \ldots + \frac{(k_0-1)!}{r_2 (r_1)^{k_0-1}}\right)|f^j_{I_*}(\mathring{x} + I_* \mathring{y})|$$
$$= \frac{1}{(r_2)^{k_0}}|f_{I_*}(\mathring{x} + I_* \mathring{y})|\left(\frac{|f_{I_*}(x_{*0} + I_* y_{*0})|}{|f_{I_*}(\mathring{x} + I_* \mathring{y})|} - \sum_{j=0}^{k_0-1} j!\left(\frac{r_2}{r_1}\right)^j\right). \tag{3.11}$$

In view of (3.4), we have $|f_{I_*}(x_{*0} + I_* y_{*0})|/|f_{I_*}(\mathring{x} + I_* \mathring{y})| > P$. Besides, this inequality holds

$$\sum_{j=0}^{k_0-1} j!\left(\frac{r_2}{r_1}\right)^j \le k_0!\left(\frac{(r_2/r_1)^{k_0} - 1}{r_2/r_1 - 1}\right) \le n_0!\frac{r_1}{r_2 - r_1}\left(\frac{r_2}{r_1}\right)^{n_0}.$$

Applying (3.7), we obtain

$$\frac{|f_{I_*}(x_{*0} + I_* y_{*0})|}{|f_{I_*}(\mathring{x} + I_* \mathring{y})|} - \sum_{j=0}^{k_0-1} j!\frac{r_2^j}{r_1^j} > P - \frac{n_0! r_1}{r_2 - r_1}\left(\frac{r_2}{r_1}\right)^{n_0} = n_0!\left(\frac{r_2}{r_1}\right)^{n_0} P_0 + 1 > 1.$$

It follows from (3.11), (3.6) and (3.9) that

$$\left|f^{(k_0)}_{I_*}(x_{*k_0} + I_* y_{*k_0})\right| > \frac{1}{(r_2)^{k_0}}\left(P - n_0!\frac{r_1}{r_2 - r_1}\left(\frac{r_2}{r_1}\right)^{n_0}\right)(r_1)^{k_0}\frac{|f^{(k_0)}_{I_*}(x_* + I_* y_*)|}{k_0!}$$
$$\ge \left(\frac{r_1}{r_2}\right)^{n_0}\left(P - n_0!\frac{r_1}{r_2 - r_1}\left(\frac{r_2}{r_1}\right)^{n_0}\right)\frac{|f^{(k_0)}_{I_*}(x_{*k_0} + I_* y_{*k_0})|}{n_0! P_1}.$$

Hence, $P < n_0!\left(\frac{r_2}{r_1}\right)^{n_0}\left(P_1 + \frac{r_1}{r_2 - r_1}\right)$, which contradicts (3.7).

Sufficiency. We choose any two numbers $r_1 \in (0,1)$ and $r_2 \in (1,+\infty)$. For given $q_0 = x_0 + I_0 y_0 \in \mathbb{H}$ we expand the function $f_I(x+Iy)$ in a power series by powers $x+Iy$

$$f_I(x+Iy) = \sum_{m=0}^{\infty} b_m(x_0+Iy_0)(x+Iy-(x_0+Iy_0))^m, \quad b_m(x_0+Iy_0) = \frac{f_I^{(m)}(x_0+Iy_0)}{m!}.$$

Since f is a slice entire regular function, the series uniformly converge in any disc $\{x+Iy : (x-x_0)^2+(y-y_0)^2 \le r^2\}$. For $r>0$, we denote

$$\begin{aligned} M(r,q_0,f) &= \max\{|f_I(x+Iy)| : (x-x_0)^2+(y-y_0)^2 = r^2\}, \\ \mu(r,q_0,f) &= \max\{|b_m(q_0)|r^m : m \ge 0\}, \\ \nu(r,q_0,f) &= \max\{m : |b_m(q_0)|r^m = \mu(r,q_0,f)\}. \end{aligned}$$

By Cauchy's inequality (3.8) one has

$$\begin{aligned} \mu(r,q_0,f) &= \max\{|b_m(q_0)|r^m : m \ge 0\} \\ &= \max\left\{\frac{|f_I^{(m)}(x_0+Iy_0)|}{m!} r^m : m \ge 0\right\} \le M(r,q_0,f). \end{aligned}$$

But for any $r>0$, we have

$$M(r_1 r, q_0, f) \le \sum_{m=0}^{\infty} |b_m(q_0)| r^m r_1^m \le \mu(r,q_0,f) \sum_{m=0}^{\infty} r_1^m = \frac{\mu(r,q_0,f)}{1-r_1}$$

and since $\nu(r,q^0,f)$ is monotone in r, we deduce

$$\ln\mu(r_2 r,q_0,f) - \ln\mu(r,q_0,f) = \int_r^{r_2 r} \frac{\nu(t,q_0,f)}{t}dt \ge \nu(r,q_0,f)\ln r_2.$$

Hence,

$$\begin{aligned} \nu(r,q_0,f) \le& \frac{1}{\ln r_2}(\ln\mu(r_2 r,q_0,f) - \ln\mu(r,q_0,f)) \\ \le& \frac{1}{\ln r_2}\{\ln M(r_2 r,q_0,f) - \ln((1-r_1)M(r_1 r,q_0,f))\} \\ =& -\frac{\ln(1-r_1)}{\ln r_2} + \frac{1}{\ln r_2}\{\ln M(r_2 r,q_0,f) - \ln M(r_1 r,q_0,f)\} \end{aligned} \tag{3.12}$$

Let $N(q_0,f)$ be the index of the function f at the point q_0, i.e., $N(q^0,f)$ is the smallest number m_0 for which inequality (2.1) holds with $q=q_0$. It is obvious that

$$N(q_0,f) \le \nu(1,q_0,f).$$

However, inequality (2.2) can be written in the following form

$$M(r_2,q_0,f) \le P_1(r_1,r_2) M(r_1,q_0,f).$$

Thus, from (3.12) for $r=1$ we obtain

$$N(q^0,f) \le -\frac{\ln(1-r_1)}{\ln r_2} + \frac{\ln P_1(r_1,r_2)}{\ln r_2}$$

for every $q_0 \in \mathbb{H}$, i.e.,

$$N(f) \le -\frac{\ln(1-r_1)}{\ln r_2} + \frac{\ln P_1(r_1,r_2)}{\ln r_2}.$$

The obtained bound does not depend on the point q_0. Thus, it justifies the definition of the global index. This completes the proof of Theorem 3.3. □

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